PMM U.S.S.R.,Vol.44,pp.342-349 0021-8928/81/3 0342 $7.50/0
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UDC 539.3

ON THE FLEXURE OF A STRIP WITH A TRANSVERSE SERIES OF CIRCULAR HOLES OR ABSOLUTELY RIGID INCLUSIONS*

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The flexure of an isotropic strip with a periodic transverse series of circular holes of rigid inclusions is considered. The state of stress is described by the functions $\varphi(z)$ and $\psi(z)$ which are analytic in the domain Ω^e occupied by the strip, and comprised of main $(\varphi^{(0)}, \psi^{(0)})$ corresponding to the solid strip, and perturbed $(\varphi^{(1)}, \psi^{(1)})$ functions /1—3/. The required functions $\varphi^{(1)}(z)$ and $\psi^{(1)}(z)$ should satisfy special conditions on the hole of inclusion outlines and on the straight line edges.

A similar problem for a single hole or inclusion was studied in /2,4/, where it was considered that the influence of the edge on the perturbed state could be neglected. Certain boundary conditions on the rectilinear edges can be satisfied strictly by going over to a computation of an infinite plate S of periodic structure whose load is converted by a one-dimensional representation of the group $C_s\,(C_{2v}{}^1, D_{2h}{}^1)$.

Direct extension of the method mentioned to the analysis of a strip with a regular transverse series of holes or inclusions is quite awkward since it is associated with the investigation of boundary conditions on a large number of outlines. However, a given load can always be decomposed into components for which significant simplifications of the solution hold and then the superposition principle can be used. Such components are the loads $Q_{\alpha\eta}$ $(\eta = 1, 2)$ which are converted by the two-dimensional representations τ_α of the symmetry group C_s of the plate S.

In connection with this investigation scheme /5/, formulas are obtained for practical application in this paper, of the decomposition of the fundamental state of stress, and an algorithm is constructed for the solution of the generalized periodic flexure problem, i.e., an analysis of a plate S with a symmetry group C_s for the loads $Q_{\alpha\eta}$ $(\eta = 1, 2)$. The appropriate functions $\varphi_{\alpha\eta}^{(1)}(z)$ and $\psi_{\alpha\eta}^{(1)}(z)$ are expressed in terms of the functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)\,(\rho = 1, 2)$ which are analytic on the exterior of the main outline and are determined from a system of its boundary conditions corresponding to each of the loads $Q_{\alpha\eta}(\eta = 1, 2)$. The numerical solution of the generalized periodic flexure problem is based on its reduction to an infinite system of algebraic equations. The state of stress of a simply supported strip is investigated during its cylindrical flexure by moments and transverse forces by using the method proposed.

1. Generalized Periodic Flexure Problem. Let us present certain information referring to the formulation and solution of the generalized periodic problem of the flexure of an infinite plate S with a regular series of circular holes or circular rigid inclusions (Fig.1).

Irreducible representations of the group C_s /6/. The translations T_r $(r = 0, \pm 1, \pm 2, \ldots)$, shifts by a vector $2rl$ along the x axis, where $2l$ is the basis vector, and reflections Θ_r $(r = 0, \pm 1, \pm 2, \ldots)$ on the planes Π_r are elements of this group, i.e., symmetry elements of the plate S. They can be multiplied as arbitrary motions. In particular

$$T_rT_m = T_{r+m},\quad T_r\Theta_m = \Theta_{m+r},\quad \Theta_mT_r = \Theta_{m-r},\quad \Theta_r\Theta_m = T_{r-m},\quad (r, m = 0, \pm 1, \pm 2, \ldots)$$

For fixed α the set of matrices

$$\tau_\alpha(T_r) = \begin{Vmatrix} \cos r\alpha & \sin r\alpha \\ -\sin r\alpha & \cos r\alpha \end{Vmatrix} \tag{1.1}$$

$$\tau_\alpha(\Theta_r) = \begin{Vmatrix} \cos r\alpha & -\sin r\alpha \\ -\sin r\alpha & -\cos r\alpha \end{Vmatrix} \quad (r = 0, \pm 1, \ldots)$$

is a representation τ_α of the group C_s since it satisfies the condition $\tau_\alpha(g_1)\tau_\alpha(g_2) = \tau_\alpha(g_1g_2)$ $(\forall g_1, g_2 \in C_s)$. The dimensionality of the representation τ_α, i.e., the order of the square matrices forming it, is two. For $0 \leqslant \alpha \leqslant \pi$ all the representations τ_α are distinct (non-equivalent). For $0 < \alpha < \pi$ the representations τ_α are irreducible and denoted by $\tau_{\alpha 1}$. In cases when $\alpha = 0, \pi$ the matrices $\tau_\alpha(g)$ are diagonal. Their upper and lower diagonal elements form the one-dimensional irreducible representations τ_{α_1} and τ_{α_2}, respectively. If

*Prikl.Matem.Mekhan.,44,No.3,485-496,1980

the number of distinct irreducible representations $\tau_{\alpha\nu}$ corresponding to the subscript α is denoted by l_α and their dimensionality by m_α, then for $\alpha = 0, \pi$ and $0 < \alpha < \pi$ the equalities $l_\alpha = 2$, $m_\alpha = 1$, and $l_\alpha = 1$, $m_\alpha = 2$ hold, respectively. Henceforth, $\tau_{\alpha\nu\rho\mu}(g)$ (ρ, $\mu = 1, 2, \ldots, m_\alpha$) and $\tau_{\alpha\rho\mu}(g)$ (ρ, $\mu = 1, 2$) are understood to be the $\rho\mu$-th elements of the matrices $\tau_{\alpha\nu}(g)$ and $\tau_\alpha(g)$.

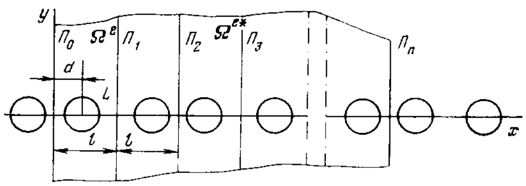

Fig.1

Functions converting into irreducible representations. The domain Ω occupied by the plate S is divided into elementary cells by reflection planes Θ_r. The main cell S^e, bounded on the left by the y axis, occupies the domain Ω^e.

Let us examine the m_α functions $f_{\alpha\nu\mu}$ ($\mu = 1, 2, \ldots, m_\alpha$) given on Ω and possessing the following properties:

$$f_{\alpha\nu\mu}(gz) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g) f_{\alpha\nu\rho}(z), \quad \forall g \in C_s, \quad \forall z \in \Omega \tag{1.2}$$

It is said about the function $f_{\alpha\nu\mu}$ that it is converted by the irreducible representation $\tau_{\alpha\nu}$ as the μ-th basis function. The load functions or the components of the state of stress-strain, whose values depend on the selection of the coordinate axes, should hence be written first in an invariant reference system requiring the introduction of local coordinates for each elementary cell. If the x and y axes are used as such in the subdomain Ω^e, then the shifted axes gx and gy are such in the subdomain $\Omega^g = g\Omega^e$. About the state of stress-strain whose components are converted as the μ-th basis functions by the representation $\tau_{\alpha\nu}$ in the invariant reference system, that it is also converted by this representation.

It is useful to simplify and unify the general symbolism for the group C_s. Let us note that for any α, one of the subscripts ν or μ is fixed (equal to one), while the other takes on the value 1 or 2, and we omit the fixed subscript. Thus, a function being converted by the representation $\tau_{\alpha\nu}$ as the ρ-th basis is denoted by $f_{\alpha\mu}$, where $\mu = \nu$ ($\alpha = 0, \pi$) or $\mu = \rho$ ($0 < \alpha < \pi$). For $\alpha = 0, \pi$ the functions $f_{\alpha\mu}$ ($\mu = 1, 2$) are not interrelated, whereupon by studying one of them it can be provisionally assumed that the other is identically zero. We then have in place of (1.2)

$$f_{\alpha\mu}(gz) = \sum_{\rho=1}^{2} \tau_{\alpha\mu\rho}(g) f_{\alpha\rho}(z), \quad \forall g \in C_s, \quad \forall z \in \Omega^e \; (\mu = 1, 2) \tag{1.3}$$

The problem of flexure of the plate S under the load $f_{\alpha\mu}$ which is being converted by the irreducible representation $\tau_{\alpha\nu}$ of the group C_s, is called a generalized periodic problem. The ordinary periodic problem is a particular case since the periodic load is a function f_{01} (or f_{02}) which is being converted by the one-dimensional irreducible representation τ_{01} (τ_{02}).

Complex Kolosov—Muskhelishvili functions. The internal force factors in the plate S are determined by using two complex functions $\varphi(z)$ and $\psi_*(z)$ which are analytic in the domain Ω /2,3/:

$$M_x + M_y = -4(1+\nu)\,\mathrm{Re}\,\varphi'(z), \quad M_y - M_x + 2iM_{xy} = 2(1-\nu)[\bar{z}\varphi''(z) + \psi_*'(z)], \quad N_x - iN_y = -4\varphi''(z) \tag{1.4}$$

where ν is the Poisson's ratio. The Sherman function $\psi(z) = \psi_*(z) + z\varphi'(z)$ is latter used in place of the function $\psi_{\alpha\mu}(z)$.

The class of functions $\varphi(z)$ and $\psi(z)$ is successfully narrowed considerably in the ordinary periodic problem by using the periodicity of the state of stress. Analogous simplifications hold in the generalized periodic problem. The fact is that a state of stress-strain of the plate S is transformed in exactly the same way by an irreducible representation $\tau_{\alpha\nu}$ of the group C_s (*) corresponding to a load $Q_{\alpha\mu}$. Let $\varphi_{\alpha\mu}(z)$ and $\psi_{\alpha\mu}(z)$ denote complex functions describing the state mentioned. Any function of this class, which is characterized by the

*) This property is investigated in the paper: Buryshkin, M. L., On application of the theory of discrete group representations in problems of equilibrium and small oscillations of linear elastic systems, VINITI, No. 208-75, (1975).

fact that the internal force factors possess the property (1.3), can be written as follows /5/:

$$\begin{matrix}\varphi_{\alpha\mu}(z)\\ \psi_{\alpha\mu}(z)\end{matrix} = \lim_{N\to\infty}\sum_{r=-N}^{N}\sum_{\rho=1}^{2}\times\left[\tau_{\alpha\rho\mu}(T_r)\begin{matrix}\Phi^{(\rho)}(z+2rl)\\ \Psi^{(\rho)}(z+2rl)\end{matrix} - \tau_{\alpha\rho\mu}(\Theta_r)\begin{matrix}\overline{\Phi^{(\rho)}}(-z+2rl)\\ \overline{\Psi^{(\rho)}}(-z+2rl)\end{matrix}\right] \quad (1.5)$$

where $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ $(\rho = 1, 2)$ are certain functions which are analytic on the exterior of the main hole with contour L. According to (1.4) and (1.5), these functions are again the desired functions in the generalized periodic problem.

Sufficient contour conditions /5/. In general, the functions $\varphi_{\alpha\mu}(z)$ and $\psi_{\alpha\mu}(z)$ should satisfy the boundary conditions on any of the contours of the infinite series. However, writing the form (1.5) permits one to be limited to satisfying a system of boundary conditions for each of the loads $Q_{\alpha\eta}$ $(\eta = 1, 2)$ on the main contour

$$K_1\varphi_{\alpha\eta}(t) + K_2[(t-\bar t)\overline{\varphi_{\alpha\eta}'(t)} + \overline{\psi_{\alpha\eta}(t)}] = f_{\alpha\eta}^{(1)}(t) + ic_{\alpha\eta}t, \quad (\eta = 1, 2) \quad (1.6)$$

where t is a point of the contour L. K_1 and K_2 are constants dependent on the kind of problem, $f_{\alpha\eta}^{(1)}(t)$ is a function related to the load $Q_{\alpha\eta}(t)$ of the contour L, and $c_{\alpha\eta}$ is a real constant determined from the condition that the deflection function be single-valued for a complete traversal around this contour. By virtue of the properties (1.3), the boundary conditions on the remaining contours are automatically satisfied.

By separating the state of stress of the plate into the main $(\varphi^{(0)}, \psi^{(0)})$ and perturbed $(\varphi^{(1)}, \psi^{(1)})$ functions, and referring the multivalued components of the complex functions to $\varphi^{(0)}(z)$ and $\psi^{(0)}(z)$, we obtain in place of (1.6)

$$K_1\varphi_{\alpha\eta}^{(1)}(t) + K_2[(t-\bar t)\overline{\varphi_{\alpha\eta}^{(1)\prime}(t)} + \overline{\psi_{\alpha\eta}(t)}] = f_{\alpha\eta}(t) \quad (\eta = 1, 2) \quad (1.7)$$

$$f_{\alpha\eta}(t) = f_{\alpha\eta}^{(1)}(t) - K_1\varphi_{\alpha\eta}^{(0)}(t) - K_2[(t-\bar t)\overline{\varphi_{\alpha\eta}^{(0)\prime}(t)} + \overline{\psi_{\alpha\eta}^{(0)}(t)}] + ic_{\alpha\eta}t \quad (1.8)$$

where $\varphi_{\alpha\eta}^{(1)}(z)$ and $\psi_{\alpha\eta}^{(1)}(z)$ are holomorphic in the domain occupied by the plate. The determination of these functions from (1.7) is the main stage in the solution of the generalized periodic problem.

The series method. Let us first note that according to the above the complete and fundamental states of stress should be converted by means of the irreducible representation $\tau_{\alpha\nu}$. Therefore, the perturbed state possesses the same property, and expressions of the type (1.5) are valid for the components $\varphi_{\alpha\eta}^{(1)}(z)$ and $\psi_{\alpha\eta}^{(1)}(z)$, where the functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ are holomorphic outside the main cavity and can be represented in the form

$$\Phi^{(\rho)}(z) = \sum_{k=1}^{\infty} a_{\rho k}(z-d)^{-k}, \quad \Psi^{(\rho)}(z) = \sum_{k=1}^{\infty} b_{\rho k}(z-d)^{-k} \quad (1.9)$$

where d is the spacing between the center of the main contour and the y-axis, and $a_{\rho k}$ and $b_{\rho k}$ $(\rho = 1, 2;\ k = 1, 2, \ldots)$ are constants.

The solution of the ordinary periodic flexure problem in series is elucidated in detail in /3/. The scheme of this solution is conserved for the extenstion being considered and is the following: Expressions of the type (1.5) for the functions $\varphi_{\alpha\eta}^{(1)}(z)$ and $\psi_{\alpha\eta}^{(1)}(z)$, together with the expansions (1.9), are substituted into (1.7). Both sides of the latter are combined on the main contour L into a Fourier series (in the powers of $\sigma = t - d = e^{i\theta}$). By equating coefficients of identical powers of σ, an infinite system of algebraic equations is formed from which the desired quantities $a_{\rho k}$ and $b_{\rho k}$ $(\rho = 1, 2;\ k = 1, 2, \ldots)$ are evaluated. For loads symmetric to the x-axis these quantities are real, the constant is $c_{\alpha\eta} = 0$, and the system itself has the form

$$\sum_{\rho=1}^{2}\sum_{k=1}^{\infty} A_{mk}^{(\eta\rho)} a_{\rho k} = -\gamma_{\eta,-m} + \sum_{\varkappa=1}^{2}\sum_{p=1}^{\infty}\gamma_{\varkappa p}C_{p+m-1}^{m}\varepsilon^{p+m}\lambda_{pm}^{(\eta\varkappa)} \quad (1.10)$$

$$b_{\eta m} = \frac{\gamma_{\eta m}}{K_2} - \left(\frac{K_1}{K_2} + \delta_{m1}\right)\sum_{\varkappa=1}^{2}\sum_{p=1}^{\infty} a_{\varkappa p}C_{p+m-1}^{m}\varepsilon^{p+m}\lambda_{pm}^{(\eta\varkappa)} - ma_{\eta m} + (1-\delta_{m1})(m-2)a_{\eta,m-2} \quad (1.11)$$

Here

$$A_{mk}^{(\eta\rho)} = -K_1\delta_{mk}\delta_{\eta\rho} + K_2(A_{mk}^{(\eta\rho)(1)} + A_{mk}^{(\eta\rho)(2)} + A_{mk}^{(\eta\rho)(3)}), \qquad A_{mk}^{(\eta\rho)(1)} = (m+k)C_{k+m-1}^{m}\varepsilon^{k+m}\lambda_{km}^{(\eta\rho)}$$

$$A_{mk}^{(\eta\rho)(2)} = -\varepsilon^{k+m+2}\lambda_{k,m+2}^{(\eta\rho)}[(m+2)C_{k+m+1}^{m+2} + kC_{k+m+1}^{m}], \qquad A_{mk}^{(\eta\rho)(3)} = \sum_{\varkappa=1}^{2}\sum_{p=1}^{\infty}\left(\frac{K_1}{K_2} + \delta_{p1}\right)C_{k+p-1}^{p}C_{p+m-1}^{m}\varepsilon^{k+2p+m}\lambda_{kp}^{(\varkappa\rho)}\lambda_{pm}^{(\eta\varkappa)}$$

$$\lambda_{km}^{(\eta\rho)} = (-1)^k\lim_{N\to\infty}\sum_{r=-N}^{N}{}^{*}\frac{\tau_{\alpha\eta\rho}(T_r)}{r^{k+m}} - \lim_{N\to\infty}\sum_{r=-N}^{N}\frac{\tau_{\alpha\eta\rho}(\Theta_r)}{(r-\varepsilon_1)^{k+m}}, \qquad \varepsilon = \frac{1}{2l}, \quad \varepsilon_1 = \frac{d}{l}$$

δ_{mk} is the Kronecker delta, C_k^m is the number of combinations of k elements taken m at a time, $\gamma_{\eta m}$ ($\eta = 1, 2$; $m = 0, \pm 1, \ldots$) are coefficients of the expansion of $f_{\alpha\eta}(t)$ in a Fourier series, and the asterisk above the summation sign by r means the component corresponding to the value $r = 0$ is missing. It can be shown that the solution of the system (1.10) by the truncation method is correct. For $\varepsilon_1 = 0.5$ and $\alpha = 0$, equations (1.10) decompose into two independent sets, one of which agrees with the known system of equations for the ordinary periodic problem /3/.

Let us note that the error of the method diminishes as the order of truncation of the system (1.10) grows. The error in the verifying contour relationships $M_r = 0$ (cavity) and $M_\theta = \nu M_r$ (inclusion is an indirect but practically acceptable estimate. For the numerical examples presented).n this paper, the order of the truncated system was taken at 40 ($k, m = 1, 2, \ldots, 20$). The errors in the verifying relationships in all the points considered did not exceed 3% of the external load intensity. As regards the corresponding verifying equalities $M_x = 0$ for the unloaded and $M_x = M$ for the loaded rectilinear edges, their accuracy increases as the number N grows in the calculation of the complex potentials by (1.5) and the parameters $\lambda_{km}^{(\eta p)}$. This number was chosen so that the relative error in the relationships mentioned did not exceed 0.5%.

2. **Scheme for computing the state of stress of the strip.** Let us examine the main steps in this scheme for a strip with a regular transverse series of n circular cavities or inclusions.

Transition to the computation of an infinite plate. The planes Π_r ($r = 0, \pm n, \ldots$) separate the infinite plate into strips of width $l^* = nl$. These strips are elementary cells in the sense of the group C_s^*, which differs from the group C_s just by the fundamental translation vector $2l^*$. We assume that the load $Q_{\beta\mu}^*$ on the plate S is converted by one of the irreducible representations of the group C_s^*. Then certain boundary conditions /6/ are automatically reproduced in the planes Π_0 and Π_n. Let us list the most important of thes?: a) simple support on two edges of the strip is the load Q_{02}^*; b) simply dependent edge (fixing the angles of rotation around these edges) is the load Q_{01}^*; c) simply supported left and simply dependent right edge is the load $Q_{\pi 2}^*$.

Let one of these conditions hold on the rectilinear edges of the strip. Then an infinite plate S whose load is converted by the appropriate irreducible representation of the group C_s^* and agrees with the given load on the strip in the cell S^{e*} located between the plates Π_0 and Π_n, can be studied instead. The state of stress-strain of the strip and the cell S^{e*} of the plate S are identical (*).

Let us note that the facts elucidated here acquire an obvious mechanical meaning if the symmetric properties of the functions $Q_{\beta\mu}^*$ are clarified by using the relationships (1.3). Thus, for instance, for a load Q_{01}^* on the plate S, the loads on adjacent strips (cells) are obtained from each other by reflection in the common boundary plane, and for the load Q_{02}^* by the same reflection with a subsequent change in sign.

Expansion of the load /6/. Thus the load $Q_{\beta\mu}^*$ of the plate S which possesses the symmetry group C_s is converted by the representation $\tau_{\beta\nu}^*$ of its subgroup C_s^*. The following expansion then holds

$$Q_{\beta\mu}^* = \sum_{\alpha \in K_\beta} Q_{\alpha\mu} \tag{2.1}$$

where K_β is the set of all numbers α of different absolute value that satisfy the relationships

$$\alpha = (\beta + 2\pi j) / n \quad (j = 0, \pm 1, \pm 2, \ldots), \ |\alpha| \leqslant \pi \tag{2.2}$$

Formulas (2.1) and (2.2) determine the structure of the decomposition of the problem into generalized periodic problems. The load components in the expansion (2.1), and the functions being converted with them by the irreducible representation $\tau_{\alpha\nu}$ of the group C_s are found from the expressions

$$Q_{\alpha\eta}(z) = \frac{m_\alpha}{m_\beta n} \sum_{\mu=1}^{2} \sum_{r=0}^{n-1} \tau_{\alpha\eta\mu}(T_r)\, Q_{\beta\mu}^*(z - 2rl), \quad \forall z \in \Omega \quad (\eta = 1, 2) \tag{2.3}$$

which have the following simple form for one-dimensional representations of the group C_s^* ($\beta = 0, \pi$)

$$Q_{\alpha\eta}(z) = \frac{m_\alpha}{n} \sum_{r=0}^{n-1} \tau_{\alpha\eta\mu}(T_r)\, Q_{\beta\mu}^*(z - 2rl), \quad \forall z \in \Omega \quad (\eta = 1, 2) \tag{2.4}$$

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*) The appropriate theorem is formulated without proof in /6/. The proof is presented in the paper mentioned in the previous footnote.

Solution of the generalized problems and their superposition. In order that the solution of the generalized problem corresponding to the load $Q_{\alpha\eta}$ be present, all the loads $Q_{\alpha\eta} = Q_{\alpha\eta}^{(0)} + Q_{\alpha\eta}^{(1)}$ $(\eta = 1, 2)$ should be determined by means of (2.3) or (2.4), where $Q_{\alpha\eta}^{(1)}$ and $Q_{\alpha\eta}^{(0)}$ are loads distributed along the hole contours and along the surface of the plate S, respectively. Finding the components $Q_{\alpha\eta}^{(1)}$ permits evaluation of the functions $f_{\alpha\eta}^{(1)}(t)$ by means of known relationships /2,3/, and the components $Q_{\alpha\eta}^{(0)}$ the evaluation of the functions $\varphi_{\alpha\eta}^{(0)}(t)$ and $\psi_{\alpha\eta}^{(0)}(t)$ from (1.8). It is quite important that it is here sufficient to have the value of the load $Q_{\alpha\eta}^{(1)}$ only on the main contour and of the functions $\varphi^{(0)}(z)$ and $\psi^{(0)}(z)$ only in the elementary cell S^e. This significantly facilitates the practical utilization of (2.3) and (2.4).

After having determined the coefficients $a_{\rho k}$ and $b_{\rho k}$ from (1.10) and (1.11) by using (1.9), (1.5) and (1.4), the components of the state of stress of the generalized periodic problem are found. Such a method of calculating the components mentioned turns out to be logical only at points of the domain Ω^e. For points in the other cells these calculations should be performed on the basis of (1.3).

The desired state of stress of the strip is constructed as the sum of the states in the generalized periodic problems generated by the expansion (2.1).

Example. To illustrate the scheme elucidated, we examine more specifically the computation of the state of stress of a strip weakened by a transverse series of n holes whose outlines are loaded by uniformly distributed bending moments of intensity M. The left edge of the strip is simply supported, while the right edge is simply dependent.

As has already been noted, it is expedient to investigate an infinite plate S with a regular series of holes instead. Let us successively number all the holes to the right of the plane Π_0 by $0, 1, 2, \ldots,$ and to the left of Π_0 by $-1, -2, \ldots$. The geometric parameters l and d characterizing the mutual location of the holes, as well as the loads on the outlines in the domain Ω^{e*} agree at the strip and the plate S. The load on the plate should be converted by the irreducible representation $\tau_{\pi 2}{}^*$ of the group $C_s{}^*$. In conformity with (1.1) and (1.3), this means that

$$M_{\pi 2}^{(r)*} = \begin{cases} M \ (r = 4Nn,\ 4Nn + 1, \ldots, 4Nn + 2n - 1) \\ -M \ (r = 4Nn + 2n,\ 4Nn + 2n + 1,\ 4Nn + 4n - 1), \quad (N = 0, \pm 1, \pm 2, \ldots) \end{cases} \tag{2.5}$$

Here $M_{\beta\mu}^{(r)*}$ and $M_{\alpha\eta}^{(r)}$ denote the intensities of the moments on the outline of the r-th hole corresponding to the loads $Q_{\beta\mu}{}^*$ and $Q_{\alpha\eta}$.

It follows from (2.2) and (2.1) that

$$Q_{\pi 2}^* = \sum_{j=1}^{R_1} Q_{(2j-1)\pi/n,\ 2} \tag{2.6}$$

Here $R_1 = n/2$ for even and $R_1 = (n+1)/2$ for odd n. Moreover, by virtue of (1.1), (2.4), and (2.5)

$$M_{(2j-1)\pi/n,\ 1}^{(0)} = -M \sum_{r=1}^{n-1} \sin \frac{(2j-1)\pi r}{n}, \qquad M_{(2j-1)\pi/n,\ 2}^{(0)} = M\left(1 - \sum_{r=1}^{n-1} \cos \frac{(2j-1)\pi r}{n}\right) \tag{2.7}$$

In conformity with the expansion (2.6), first the R_1 of the generalized periodic problems are solved. For the j-th problem we must put in (1.7), (1.8) and (1.10)

$$K_1 = (3+\nu)/(1-\nu), \quad K_2 = -1, \quad \varphi_{\alpha\eta}^{(0)}(z) = \psi_{\alpha\eta}^{(0)}(z) = 0, \qquad f_{(2j-1)\pi/n,\ \eta}^{(1)}(t) = \gamma_{\eta j}\, \sigma \ (\eta = 1, 2)$$

$$\gamma_{11} = -M \sum_{r=1}^{n-1} \sin \frac{(2j-1)\pi r}{n}, \quad \gamma_{21} = M\left(1 - \sum_{r=1}^{n-1} \cos \frac{(2j-1)\pi r}{n}\right)$$

and $\gamma_{\mu m} = 0$ $(m \neq 1)$. The components of the states of stress obtained in the generalized problems are superposed by formulas identical to (2.6).

Values of $M_\theta{}^* = M_\theta / M$ and $M_r{}^* = M_r / M$ are presented at points of the outlines of different holes in Table 1 for $l = 2.4$, $d = 1.2$, $\nu = 1/3$ as a function of the polar angle θ and the number n of holes in the strip. The number of the holes is indicated in the column denoted by N. The necessary data are also presented for the ordinary periodic problem $(n = \infty)$. It is seen that a state of stress similar to the periodic is built up in the zone adjoining the simply dependent edge. The state of stress near the free edge has a substantial difference. The quantitative and qualitative questions occurring in this connection are investigated in more detail in Sect.4.

3. Expansion of the main state of stress. When the load is distributed over the edges or the surface of the strip, the direct utilization of (2.3) and (2.4) is difficult. It is more convenient to decompose the state of stress of the plate S rather than the load, i.e., to decompose the functions $\varphi_{\beta\mu}{}^{(0)*}(z)$ and $\psi_{\beta\mu}{}^{(0)*}(z)$ into components $\varphi_{\alpha\eta}{}^{(0)}(z)$ and $\psi_{\alpha\eta}{}^{(0)}(z)$.

Let us first introduce the quantity R_2 which equals $n/2$ and $(n-1)/2$, respectively,

Table 1

n	N		$\theta=0$	$\pi/10$	$3\pi/10$	$\pi/2$	$7\pi/10$	$9\pi/10$	π
2	1	M_θ^*	−1.85	−1.25	−0.65	−1.13	−1.48	−0.85	−0.48
	1	M_θ^*	−2.33	−1.59	−0.58	−1.11	−1.57	−0.87	−0.49
4		M_θ^*	−2.72	−1.90	−0.66	−0.75	−0.75	−1.69	−2.41
	2	M_r^*	0.03	0.03	0.00	−0.02	−0.01	0.01	0.02
	1	M_θ^*	−2.58	−1.75	−0.53	−1.07	−1.60	−0.87	−0.49
8	2	M_θ^*	−3.16	−2.26	−0.64	−0.76	−0.84	−1.72	−2.59
	4	M_θ^*	−3.55	−2.50	−0.74	−0.67	−0.76	−2.45	−3.47
	1	M_θ^*	−2.64	−1.81	−0.53	−1.07	−1.61	−0.87	−0.49
		M_r^*	0.02	0.02	−0.01	−0.01	0.00	0.01	0.01
16	4	M_θ^*	−3.93	−2.78	−0.75	−0.64	−0.81	−2.62	−3.70
		M_θ^*	−4.24	−3.01	−0.81	−0.61	−0.82	−3.00	−4.22
	8	M_r^*	0.03	0.02	−0.01	−0.01	−0.01	0.02	0.03
∞	—	M_θ^*	−5.16	−3.69	−0.92	−0.54	−0.92	−3.69	−5.16

for even and odd n, and let us also note that in conformity with (1.3)

$$Q_{\beta\mu}^*(z)=\sum_{\rho=1}^{2}\tau_{\beta\mu\rho}^*(\Theta_0)\,Q_{\beta\rho}^*(-\bar z),\quad Q_{\beta\mu}^*(z)=\sum_{\rho=1}^{2}\tau_{\beta\mu\rho}^*(T_{-n})\,Q_{\beta\rho}(z+2nl) \tag{3.1}$$

Moreover, it follows from (1.1) and (2.2) that

$$\tau_{\beta\mu\rho}^*(\Theta_0)=\tau_{\alpha\mu\rho}(\Theta_0),\quad \tau_{\beta\mu\rho}^*(T_{-n})=\tau_{\alpha\mu\rho}(T_{-n}) \tag{3.2}$$

from which there results by virtue of elementary properties of the representations

$$\sum_{\mu=1}^{2}\tau_{\alpha\eta\mu}(T_r)\tau_{\alpha\mu\rho}^*(\Theta_0)=\tau_{\alpha\eta\rho}(\Theta_r),\quad \sum_{\mu=1}^{2}\tau_{\alpha\eta\mu}(T_r)\tau_{\beta\mu\rho}^*(T_{-n})=\tau_{\alpha\eta\rho}(T_{r-n}) \tag{3.3}$$

Now, using the relationships (3.1)− (3.3) instead of (2.3), we obtain

$$Q_{\alpha\eta}(z)=\frac{m_\alpha}{m_\beta n}\sum_{\mu=1}^{2}\Big\{\tau_{\alpha\eta\mu}(T_0)\,Q_{\beta\mu}^*(z)+\sum_{r=1}^{R_z}\tau_{\alpha\eta\mu}(T_r)\,Q_{\beta\mu}^*(z-2rl)\;+\sum_{r=1}^{n-R_z-1}\tau_{\alpha\eta\mu}(T_{n-r})\,Q_{\beta\mu}^*(z-2nl+2rl)\Big\}= \tag{3.4}$$

$$\frac{m_\alpha}{m_\beta n}\sum_{\rho=1}^{2}\Big\{\sum_{r=1}^{R_z}\tau_{\alpha\eta\rho}(\Theta_r)\,Q_{\beta\rho}^*(-\bar z+2rl)\;+\sum_{r=0}^{n-R_z-1}\tau_{\alpha\eta\rho}(T_{-r})\,Q_{\beta\rho}^*(z+2rl)\Big\}\quad(\eta=1,2)$$

This formula is valid for any function written in the invariant reference system, including those for the bending moments and torques. Let us note that the sign of the torques should be reversed in the subdomains $\Theta_r\Omega^e$ when writing in the invariant system. In this connection, we obtain by using (1.4) and (3.4)

$$\operatorname{Re}\varphi_{\alpha\eta}'(z)=\frac{m_\alpha}{m_\beta n}\sum_{\rho=1}^{2}\Big\{\sum_{r=1}^{R_z}\tau_{\alpha\eta\rho}(\Theta_r)\operatorname{Re}\varphi_{\beta\rho}^{*\prime}(-\bar z+2rl)\;+\sum_{r=0}^{n-R_z-1}\tau_{\alpha\eta\rho}(T_{-r})\operatorname{Re}\varphi_{\beta\rho}^{*\prime}(z+2rl)\Big\}\quad(\eta=1,2) \tag{3.5}$$

If ω_0 is the neighborhood of the point z_0 lying entirely in the domain of analyticity of the complex functions, then the neighborhood $g\omega_0$ of the point gz_0 also belongs to this domain. We assume that

$$\varphi_{\alpha\eta}(z)=\sum_{k=1}^{\infty}a_k^{(\alpha\eta)}(z-z_0)^k,\quad \varphi_{\beta\rho}^*(\Theta_r z)=\sum_{k=1}^{\infty}b_{kr}^{(\beta\rho)}(\Theta_r z-\Theta_r z_0)^k$$

$$\varphi_{\beta\rho}^*(T_r z)=\sum_{k=1}^{\infty}c_{kr}^{(\beta\rho)}(T_r z-T_r z_0)^k,\quad \forall z\in\omega_0\quad(\eta=1,2) \tag{3.6}$$

The constant in the Taylor series is omitted since it does not affect the state of stress of the plate. After substituting the series (3.6) into the equality (3.5) and equating coefficients for identical powers of the difference $z-z_0$, we find

$$a_k^{(\alpha\eta)}=\frac{m_\alpha}{m_\beta n}\sum_{\rho=1}^{2}\Big\{\sum_{r=1}^{R_z}\tau_{\alpha\eta\rho}(\Theta_r)(-1)^{k-1}\overline{b_{kr}^{(\beta\rho)}}+\sum_{r=0}^{n-R_z-1}\tau_{\alpha\eta\rho}(T_{-r})\,c_{kr}^{(\beta\rho)}\Big\},\quad(k=1,2,\ldots)$$

By using this relationship, the first of (3.6) is reduced to the desired formula.

Using an analogous method and considering an appropriate representation for

$$(\bar{z}-z)\,\varphi_{\alpha\eta}''(z)-\varphi_{\alpha\eta}'(z)+\psi_{\alpha\eta}(z)$$

instead of (3.5), we obtain the required expansion $\psi_{\alpha\eta}(z)$. Both expansions have the form

$$\chi_{\alpha\eta}(z)=\frac{m_\alpha}{m_\beta n}\sum_{\rho=1}^{2}\Big\{\sum_{r=0}^{n-R_2-1}\tau_{\alpha\eta\rho}(T_{-r})\,\chi_{\beta\rho}^{*}(z+2rl)-\sum_{r=1}^{R_2}\tau_{\alpha\eta\rho}(\Theta_r)\,\overline{\chi_{\beta\rho}^{*}}(-z+2rl)\Big\}(\chi=\varphi,\psi) \tag{3.7}$$

For one-dimensional representations of the group C_s^* only the first term corresponding to the value $\rho=\mu$ is conserved in the sum over ρ.

Let us note that the complex functions $\varphi_{\beta\rho}^*(z)$ and $\psi_{\beta\rho}^*(z)$ need not be found on the whole plate S to apply the expansion (3.7). It is sufficient to limit oneself to giving them in the domain Ω^{e*} occupied by the strip since the components $\varphi_{\alpha\eta}(z)$ and $\psi_{\alpha\eta}(z)$ should be determined only in the domain Ω^e for the solution of the generalized problem.

4. State of stress of a freely supported strip. The purpose of the investigation conducted below is to refine the estimate of the state of stress of a strip of finite width for the problem of moment concentration around a regular transverse series of circular holes or inclusions. Without limiting the generality, the radius of these latter is taken equal to unity.

According to Sect.2, the state of stress of the band under consideration and of the infinite plate S in the domain Ω^{e*} is identical, if the plate load is obtained by the irreducible representation τ_{02}^* of the group C_s^*. It follows from (2.2) and (2.1) that

$$Q_{02}^{*}=\sum_{j=0}^{R_2}Q_{2j\pi/n,\,2} \tag{4.1}$$

The contour loads are considered zero, i.e., $f_{\alpha\eta}^{(1)}(t)=0$. In conformity with the recommendations of Sect.3, we decompose the main state of stress into components being converted by the irreducible representations of the group C_s^* rather than the load. In connection with (4.1), we have

$$\varphi_{02}^{(0)*}(z)=\sum_{j=0}^{R_2}\varphi_{2j\pi/n,\,2}^{(0)}(z),\quad \psi_{02}^{(0)*}(z)=\sum_{j=0}^{R_2}\psi_{2j\pi/n,\,2}^{(0)}(z) \tag{4.2}$$

where the functions $\varphi_{2j\pi/n,2}^{(0)}(z)$ and $\psi_{2j\pi/n,2}^{(0)}(z)$ are determined by (3.7) in which the functions $\varphi^{(0)}(z)$ and $\psi^{(0)}(z)$ describing the state of stress of a solid strip can be used in place of $\varphi_{02}{}^{(0)*}(z)$ and $\psi_{02}{}^{(0)*}(z)$.

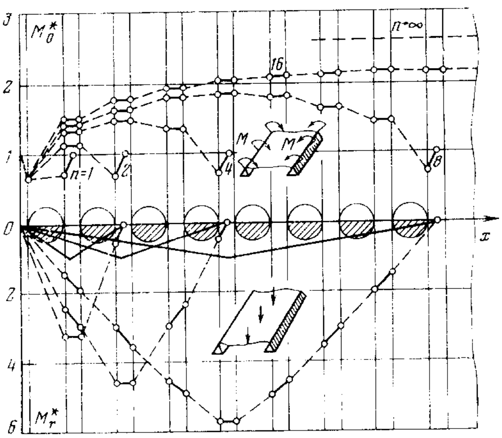

Fig.2

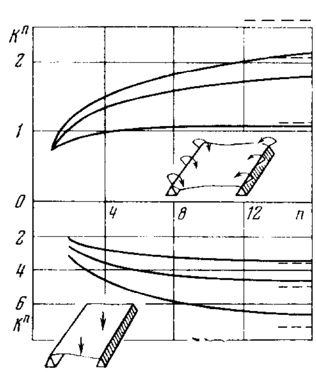

Fig.3

Pure cylindrical flexure. A strip with holes is bent by moments of intensity M which are distributed along the rectilinear edges. The main state of stress of the strip is described by the functions $\varphi^{(0)}(z)=-Mz/4$, $\psi^{(0)}(z)=-3Mz/4$. As an illustration, we present the nonzero components of the expansion (4.2) and their related functions defined to the accuracy of a constant for the case $n=8$

$$\varphi_{\pi/4,\,1}^{(0)}(z)=\frac{M}{8}(1+\sqrt{2})\,z,\quad \varphi_{\pi/4,\,2}^{(0)}(z)=-\frac{M}{8}\,z,\qquad \varphi_{3\pi/4,1}^{(0)}(z)=-\frac{M}{8}(1-\sqrt{2})\,z$$

$$\varphi^{(0)}_{3\pi/4,2}(z) = -\frac{M}{8} z, \quad \psi^{(0)}_{\alpha\eta}(z) = 3\varphi^{(0)}_{\alpha\eta}(z) \quad \left(\alpha = \frac{\pi}{4}, \frac{3\pi}{4}\right)$$

Analysis of the numerical results shows that from the strength viewpoint the bending moments M_θ are the greatest danger in the middle part of the strip at those points of the contours that are on the x axis. The quantities $M_\theta^* = M_\theta / M$ at the points mentioned are presented in the upper half of Fig.2 for $l = 2.3$ and $\varepsilon_1 = 0.5$. To be more graphic, the constructed points of the graphs are provisionally connected by continuous segments to the holes by necks and shading. It is seen that for sufficiently large n a state of stress close to the periodic kind occurs at the middle part of the strip. In this zone the maximum moment concentration occurs (the level of the moments of the main state of stress is shown by the continuous lines).

Table 2

n	N	$\theta=0$	$\pi/10$	$3\pi/10$	$\pi/2$	$7\pi/10$	$9\pi/10$	π
2	1	1.16	0.92	1.27	1.99	1.94	1.08	0.70
	1	1.35	1.05	1.23	1.98	1.98	1.09	0.70
4	2	1.49	1.14	1.18	1.62	1.22	1.06	1.37
	1	1.45	1.10	1.19	1.94	1.97	1.08	0.70
8	2	1.70	1.27	1.16	1.62	1.27	1.13	1.45
	4	1.84	1.37	1.17	1.53	1.18	1.35	1.81
	1	1.48	1.12	1.18	1.94	1.98	1.08	0.70
	4	2.03	1.49	1.17	1.52	1.20	1.43	1.91
16	6	2.12	1.56	1.18	1.49	1.19	1.54	2.08
	8	2.15	1.59	1.18	1.48	1.18	1.58	2.14
∞	—	2.60	1.90	1.21	1.43	1.21	1.90	2.60

An additional representation of the nature of the moment distribution in a strip with the mentioned geometric characteristics is given by Table 2 in which the quantities M_θ / M at points of different contours are presented as a function of the polar angle θ and the number n.

Graphs of the dependence of the maximum moment concentration coefficient K^n for the strip at points of the x axis on the number of holes for $\varepsilon_1 = 0.5$ and different values of l are given in the upper half of Fig.3. For an infinite number of holes, i.e., for the ordinary periodic problem, the coefficient K^∞ (the dashed lines) always turn out to be higher than for the finite number. For $n > 8$ and $l > 3$ it can be considered that $K^n \approx K^\infty$. However, such an equality is achieved for considerably larger values of n as the holes are approached. In this connection, the differences between K^n and K^∞ become perceptible for $l < 2.5$.

Cylindrical bending by forces. A strip with an even number of rigid inclusions is bent by transverse forces of intensity q distributed uniformly over the axis of the strip. Let us introduce the notation $M = qnl/4$. The provisional lines of the distribution of the quantities M_r / M_θ along the x-axis for $\varepsilon_1 = 0.5$ and $l = 2.3$, as well as graphs of the moment concentration coefficient K^n on the contour with number $n/2$ are presented in the lower half of Figs.2 and 3.

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Translated by M.D.F.